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Extension groups between simple Mackey functors

Muriel Nicollerat

EPFL, Institut de Géométrie, Algèbre et Topologie, 1015 Lausanne, Switzerland

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ABSTRACT

In order to better understand the structure of indecomposable projective Mackey functors, we study extension groups of degree 1 between simple Mackey functors. We explicitly determine these groups between simple functors indexed by distinct normal subgroups. We next study the conditions under which it is possible to restrict ourselves to that case, and we give methods for calculating extension groups between simple Mackey functors which are not indexed by normal subgroups. We then focus on the case where the simple Mackey functors are indexed by the same subgroup. In this case, the corresponding extension group can be embedded in an extension group between modules over a group algebra, and we describe the image of this embedding. In particular, we determine all extension groups between simple Mackey functors for a p-group and for a group that has a normal p-Sylow subgroup. Finally, we compute higher extension groups between simple Mackey functors for a group that has a p-Sylow subgroup of order p.

Introduction

We study extension groups of degree 1 between simple Mackey functors for a group G. The calculation of these groups is a very important tool in determining the Loewy series of an indecomposable projective Mackey functor. In [7], Thévenaz and Webb proved that Mackey functors may be seen as modules over a finite-dimensional algebra. This allows one to apply the representation theory of such algebras, such as for example the existence of projective covers. Moreover, the classification of simple Mackey functors (see [8]) gives a parametrization of the indecomposable projective Mackey functors over a field, which enabled Thévenaz and Webb to describe many properties of these projective functors. Let us cite some articles about these projective functors. In [3], Bouc deals with some standard resolutions of Mackey functors, in [9] Webb, constructs some specific filtration of projective functors.

E-mail address: muriel.chaubert@gmail.ch.

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In [10] and in [11], Yaraneri studies Clifford theory and inflation of projective functors. Finally, there is a study of the socle of projective functors in [6]. Nevertheless, there is still a lot left to understand about the structure of such Mackey functors.

We begin by explicitly calculating extension groups between simple functors indexed by distinct normal subgroups. We then give some conditions under which it is possible to restrict to this case. When this restriction is not possible, we obtain some information about extension groups using some specific Mackey functors, called functors T, whose definition is similar to those of the simple Mackey functors described in the classification. This enables us to give methods for calculating extension groups between simple Mackey functors which are not indexed by normal subgroups.

We also study the case of extension groups between simple Mackey functors indexed by the same subgroup. These groups may be embedded in an extension group between modules over group algebras. We describe the image of this embedding and we give some conditions under which it is an isomorphism. In particular, this is the case when G is a p-group or when G has a normal Sylow p-subgroup.

Extension groups of higher degree are closely related to minimal projective resolutions of simple Mackey functors. We exhibit some of these resolutions in the case of a group G whose order is not divisible by p^2, which is the case where the Mackey algebra has finite representation type. In particular, we show that every simple Mackey functor for such a group possesses a minimal projective resolution which is periodic.

Throughout this paper, G denotes a finite group and k an algebraically closed field of characteristic p. We write $H \leqslant G$ (respectively $H < G$) to indicate that H is a subgroup of G (respectively a proper subgroup of G). If H and K are subgroups of G, $K =_G H$ means that K is G-conjugate to H, and similarly we write $K \leqslant_G H$ (respectively $K <_G H$) when K is G-conjugate to a subgroup of H (respectively a proper subgroup of H). The transporter from K to H, which is $\{g \in G \mid gKg^{-1} \leqslant H\}$, is denoted $T_G(K,H)$. We denote by $[K\backslash G/H]$ an arbitrary set of representatives of double cosets KgH. Finally, for $H \leqslant G$ and $g \in G$, we write $\overline{N}_G(H) = N_G(H)/H$, ${}^gH = gHg^{-1}$ and $H^g = g^{-1}Hg$.

1. Preliminaries

We shall use the same notation and basic constructions which appear in [7]. Let us summarize the most basic ones. We will work with Mackey functors for a finite group G over an algebraically closed field k of characteristic p. If M is such a functor, it possesses operations of induction $I_K^H : M(K) \to M(H)$, of restriction $R_K^H : M(H) \to M(K)$ and of conjugation $c_g : M(H) \to M({}^gH)$ for every $K \leqslant H \leqslant G$ and $g \in G$. The category of Mackey functors for G over k is denoted $\mathrm{Mack}_k(G)$.

Mackey functors may also be seen as modules over a finite-dimensional k-algebra, denoted $\mu_k(G)$ and called the Mackey algebra (see [7], Section 3). Moreover, there exists a morphism from the Burnside ring $B(G)$ to the center of $\mu_k(G)$ (see [7], Proposition 9.2) which allows us to use idempotents of the Burnside ring to get a partial separation of Mackey functors into different blocks. These idempotents are classified by p-perfect subgroups J of G (that is subgroups J with no proper normal subgroup N with J/N a p-group) and denoted f_J. We denote by $\mathrm{Mack}_k(G,J)$ the full subcategory of $\mathrm{Mack}_k(G)$ whose objects are Mackey functors M for which $f_J \cdot M = M$.

Let us rapidly go over some constructions associated to Mackey functors. We denote by $\uparrow_H^G : \mathrm{Mack}_k(H) \to \mathrm{Mack}_k(G)$ the induction and by $\downarrow_H^G : \mathrm{Mack}_k(G) \to \mathrm{Mack}_k(H)$ the restriction of Mackey functors. More precisely, if $M \in \mathrm{Mack}_k(H)$ and if $K \leqslant G$ then the evaluation of $M\uparrow_H^G$ at K is equal to $M\uparrow_H^G(K) = \bigoplus_{g\in I} M(H \cap {}^g K)$ where $I = [H\backslash G/K]$; moreover, if $M \in \mathrm{Mack}_k(G)$ and if $K \leqslant H$ then the evaluation of $M\downarrow_H^G$ at K is equal to $M(K)$. Induction of Mackey functors is both right and left adjoint of restriction. Whenever N is a normal subgroup of G and M is a Mackey functor for G/N, we can form the inflation $\mathrm{Inf}_{G/N}^G M$ which is the Mackey functor defined by

$$\mathrm{Inf}_{G/N} M(K) = \begin{cases} 0 & \text{if } K \ngeqslant N, \\ M(K/N) & \text{otherwise.} \end{cases}$$

Inflation has a right adjoint, denoted $^+$, and a left adjoint, denoted $^-$. If M is a Mackey functor for G, these two functors are defined by $M^+(K/N) = M(K)/\sum_{J\leqslant K,\, J\ngeqslant N} I_J^K(M(J))$ and $L^-(K/N) = \bigcap_{J\leqslant K,\, J\ngeqslant N} \mathrm{Ker}(R_J^K)$ (for more details about the previous definitions and results, see [8]). Using these properties of adjunction, we obtain the following result.

Proposition 1.1. *Let M be a Mackey functor for a group G, $H \leqslant G$ and V a $k\overline{N}_G(H)$-module. If $M(K) = 0$ for every proper subgroup K of H, then there exists an isomorphism of k-vector spaces*

$$\varphi : \mathrm{Hom}_{\mu_k(G)}\big(M, \big(\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} FP_V\big)\uparrow_{N_G(H)}^G\big) \xrightarrow{\sim} \mathrm{Hom}_{k\overline{N}_G(H)}\big(M(H), V\big)$$

given by evaluation at H. Moreover, if $\sigma \in \mathrm{Hom}(M(H), V)$, then $\varphi^{-1}(\sigma) = (\overline{\sigma}(K))_{K\leqslant G}$, where $\overline{\sigma}(K) = 0$ if K contains no conjugate of H, and $\overline{\sigma}(K)(a) = (\sigma(c_x R_{H^x}^K(a)))_{x\in T}$ otherwise, where $T = [N_G(H)\backslash T_G(H,K)/K]$ and $a \in M(K)$ (using the formula of evaluation at K of an induced Mackey functor described above).

Proof. Using properties of adjunction given by Propositions 4.2, 5.1 and 6.1 of [8], we obtain the following bijections:

$$\begin{aligned}\mathrm{Hom}_{\mu_k(G)}\big(M, \big(\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} FP_V\big)\uparrow_{N_G(H)}^G\big) &\cong \mathrm{Hom}_{\mu_k(N_G(H))}\big(M\downarrow_{N_G(H)}^G, \mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} FP_V\big)\\ &\cong \mathrm{Hom}_{\mu_k(\overline{N}_G(H))}\big(\big(M\downarrow_{N_G(H)}^G\big)^+,\ FP_V\big)\\ &\cong \mathrm{Hom}_{k(\overline{N}_G(H))}\big(\big(M\downarrow_{N_G(H)}^G\big)^+(1), V\big).\end{aligned}$$

Moreover, using the hypothesis on M, one obtains

$$\big(M\downarrow_{N_G(H)}^G\big)^+(1) = M(H)/\sum_{J<H} I_J^H\big(M(J)\big) = M(H).$$

Finally, in order to determine the application φ^{-1}, we just need to follow the proofs of the above Propositions, where the adjunctions are constructed explicitly. □

Let us now go over the classification of simple Mackey functors established by Thévenaz and Webb [8, Theorem 8.3].

Proposition 1.2. *Let Ω be the set of pairs (H, V) where H is a subgroup of G and V is a simple $k\overline{N}_G(H)$-module, up to isomorphism. Let Ω/G be the set of G-orbits, for the action of G by conjugation. There is a well-defined map Φ from the set of classes of isomorphism of simple Mackey functors to the set Ω/G, given by $\Phi(S) = (H, S(H))$ where H is a minimal subgroup with $S(H) \neq 0$. Furthermore, Φ is a bijection. If $S_{H,V} = \Phi^{-1}(H, V)$, then $S_{H,V}$ is the unique minimal subfunctor of $(\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} FP_V)\uparrow_{N_G(H)}^G$ where FP_V is the fixed point functor (see [8], Section 6). Moreover, this simple subfunctor is isomorphic to $(\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} S_{1,V}^{\overline{N}_G(H)})\uparrow_{N_G(H)}^G$, where $S_{1,V}^{\overline{N}_G(H)}$ is the subfunctor of FP_V defined by $S_{1,V}^{\overline{N}_G(H)}(J) = \mathrm{Im}(I_1^J : V \to V^J)$.*

Since Mackey functors can be seen as modules over the finite-dimensional k-algebra $\mu_k(G)$, there is a bijection between simple Mackey functors and indecomposable projective Mackey functors. More precisely, every Mackey functor $S_{H,V}$ has a correspondence with its own projective cover denoted by $P_{H,V}$. A very important example of projective Mackey functors is given by the Burnside Mackey functor B^G. It is defined by $B^G(H) = B(H)$ the Burnside ring of H, that is the free k-module with basis the H-sets H/J where J is taken up to H-conjugacy, and the applications of induction, restriction and conjugation come from the corresponding ones for G-sets (for details, see [7], Section 8). Moreover, every indecomposable projective functor $P_{H,V}$ can be seen as a summand of an induced Burnside functor (see [7], Theorem 8.6):

Theorem 1.3. *The multiplicity of $P_{H,V}$ as a summand of $B^K\uparrow_K^G$ is $\dim(S_{H,V}(K))$; that is $B^K\uparrow_K^G \cong \bigoplus_{(H,V)} \dim_k(S_{H,V}(K)) \cdot P_{H,V}$, where the sum is taken over pairs (H, V) where H is a subgroup of G, up to conjugation, and V is a simple $k\overline{N}_G(H)$-module, up to isomorphism. In particular, if G is a p-group, then $S_{H,k} \cong B^H\uparrow_H^G$.*

Let us finally remark that these projective Mackey functors are stable under induction:

Proposition 1.4. *Let P be a projective Mackey functor for a subgroup H of G. Then, the induced functor $P\uparrow_H^G$ is also projective.*

Proof. One can verify that the functor $\mathrm{Hom}_{\mu_k(G)}(P\uparrow_H^G, \_)$ is exact, using the fact that the functor $\mathrm{Hom}_{\mu_k(H)}(P, \_)$ is exact and that the functor of induction is both left and right adjoint of the functor of restriction (see [8], Proposition 4.2). $\square$

2. Extension groups of degree 1

We fix a finite group G and an algebraically closed field k, of characteristic p. We want to study $\mathrm{Ext}(S_{Q,W}, S_{H,V}) := \mathrm{Ext}^1_{\mu_k(G)}(S_{Q,W}, S_{H,V})$, for any simple Mackey functors $S_{Q,W}$ and $S_{H,V}$. The calculation of these groups is indeed a very important tool in determining the Loewy series of projective Mackey functors. We begin by focusing on the case where Q and H are non-conjugate subgroups of G. The main result deals with the case where they are normal.

Theorem 2.1. *Let H and Q be normal subgroups of G with $H \neq Q$, let V be a simple $k\overline{N}_G(H)$-module and let W be a simple $kN_G(Q)/Q$-module. The k-vector space $\mathrm{Ext}(S_{Q,W}, S_{H,V})$ is isomorphic to*

$$\begin{cases} k & \text{if } H < Q,\ [Q:H] = p,\ V^{Q/H} = V \text{ and } V \cong W \text{ (as } kG/Q\text{-module)},\\ k & \text{if } Q < H,\ [H:Q] = p,\ W^{H/Q} = W \text{ and } W \cong V \text{ (as } kG/H\text{-module)},\\ 0 & \text{otherwise,} \end{cases}$$

where $V^{Q/H}$ is the set of elements of V fixed by the action of Q/H, and similarly $W^{H/Q}$ is the set of elements of W fixed by the action of H/Q.

Proof. Let us first suppose that $H < Q$. Given Theorem 14.3 of [7] the dimension over k of $\mathrm{Ext}(S_{Q,W}, S_{H,V})$ equals the multiplicity of $S_{Q,W}$ in the second socle layer of the functor $M = \mathrm{Inf}_{G/H}^G FP_V$. By Lemma 8.1 of [8], the socle of M equals $S \cong S_{H,V}$, the subfunctor of M generated by $S(H) = V = M(H)$. Moreover, as H is a normal subgroup, this subfunctor is defined by $\mathrm{Inf}_{G/H}^G S_{1,V}^{G/H}$ and $S(K) = I_H^K(V)$, for every subgroup K of G containing H. We have to determine the multiplicity of $S_{Q,W}$ in $M/\mathrm{Soc}(M) = M/S$. In order to do so, we will prove that M/S has a subfunctor isomorphic to $S_{Q,W}$ only if $[Q:H] = p$, $V^{Q/H} = V$ and $W = V$. Later on, we will see that under the three previous conditions, there exists a unique subfunctor of M/S isomorphic to $S_{Q,W}$.

Let T be a simple subfunctor of M/S, isomorphic to $S_{Q,W}$. In particular, $W = T(Q)$ is a submodule of $M(Q)/S(Q) = V^{Q/H}/I_H^Q(V)$ and $V^{Q/H}$ possesses a structure of kG/H-module due to the fact that Q is a normal subgroup. Hence, $V^{Q/H}$ is a kG/H-submodule of V. Consequently, by simplicity of V, there are two possible cases: either $V^{Q/H} = 0$, or $V^{Q/H} = V$. In the first case, there is no such subfunctor T in M/S. So, we can assume that $V^{Q/H} = V$, or in other words, that Q acts trivially on V. This implies that if $x \in V$, we have $I_H^Q(x) = [Q:H] \cdot x$. Therefore, if p does not divide $[Q:H]$, then $(M/S)(Q) = V/V = 0$ and there is again no subfunctor isomorphic to $S_{Q,W}$ in M/S.

Let us suppose then that $V^{Q/H} = V$ and that p divides $[Q:H]$. As Q acts trivially on V, this module naturally possesses a structure of simple kG/Q-module. As $W \subseteq (M/S)(Q) = V$, the only possibility is $W = V$. Let us next assume that there exists a subgroup J such that $H < J < Q$ and $p \mid [J:H]$. Hence, on the one hand, $R_J^Q(T(Q)) = V$, since the application of restriction is an inclusion. Indeed, as $S(J) = 0 = S(Q)$, the application R^Q_J of the functor T comes from those of $M = \mathrm{Inf}^G_{G/H} FP_V$, which is an inclusion by definition of the fixed point functor. However, on the other hand, $R^Q_J(T(Q)) \subseteq T(J) = 0$, since T is a subfunctor and the evaluation of T at proper subgroups of Q is zero. Therefore, in the latter case, there cannot exist such a subfunctor T; or in other words, $[Q : H]$ must be equal to p.

To sum up, we can assume that $[Q : H] = p$, $V^{Q/H} = V$ and $W = V$, because otherwise the multiplicity of $S_{Q,W}$ in M/S is zero. Under these assumptions, $(M/S)(Q) = V$. Let N be the subfunctor of M/S generated by $N(Q) = V$. We will show that N is equally a subfunctor of $\mathrm{Inf}^G_{G/Q} FP_V$. For this purpose, remark that if $J < Q$, then either $J = H$, in which case $(M/S)(J) = V/V = 0$, or $J \neq H$, in which case $M(J) = 0$ and thus $(M/S)(J) = 0$. Consequently, $N(J) = 0$ if $J < Q$. Furthermore, due to the definition of N, the module $N(J)$ is zero if the subgroup J does not contain Q (one can see this using Proposition 2.4 of [7], with χ being the set of subgroups of Q). If $Q < K$, then $S(K) = I^K_H(V) = I^K_Q(I^Q_H(V)) = 0$, and so $(M/S)(K) = M(K)$. Therefore, $N(K)$ is a submodule of $M(K) = V^{K/H} = V^{K/Q} = \mathrm{Inf}^G_{G/Q} FP_V(K)$, since Q acts trivially on V. Moreover, the applications of induction, restriction and conjugation of M/S are the same as those of the functor $\mathrm{Inf}^G_{G/Q} FP_V$ (using the definition of inflation and the fact that the functor S is zero when evaluated at the subgroups containing Q). Hence, N can be identified to the subfunctor of $\mathrm{Inf}^G_{G/Q} FP_V$ generated by $N(Q) = V$; it is thus isomorphic to $S_{Q,W}$. Consequently, if T is a simple subfunctor of M/S, isomorphic to $S_{Q,W}$, then N must be a non-zero subfunctor of T, as $T(Q) = V$ and as N is the smallest subfunctor of M/S whose evaluation at Q equals V. Since T is simple, it follows that $T = N$; or in other words, there is exactly one copy of $S_{Q,W}$ in the second socle layer of M. Thus, in this case, $\dim_k(\mathrm{Ext}(S_{Q,W}, S_{H,V})) = 1$.

Let us next assume that $Q < H$. Then

$$\mathrm{Ext}_{\mu_k(G)}(S_{Q,W}, S_{H,V}) \cong \mathrm{Ext}_{\mu_k(G)^{\mathrm{op}}}\big((S_{H,V})^\star, (S_{Q,W})^\star\big) \cong \mathrm{Ext}_{\mu_k(G)}(S_{H,V^\star}, S_{Q,W^\star})$$

using the fact that for every simple Mackey functor $S_{J,U}$, we have $(S_{J,U})^\star \cong S_{J,U^\star}$ (by Proposition 4.1 of [7]) and that $\mu_k(G)^{\mathrm{op}} \cong \mu_k(G)$ (by Chapter 4 of [7]). Using the case where $H < Q$, this k-vector space equals k if and only if $[H : Q] = p$, $(W^\star)^{H/Q} = W^\star$ and $W^\star \cong V^\star$. Yet $(W^\star)^{H/Q} = W^\star$ if and only if $W^{H/Q} = W$ and, similarly, $W^\star \cong V^\star$ if and only if $V \cong W$.

Finally, if $Q \not< H$ and $H \not< Q$, the group $\mathrm{Ext}(S_{Q,W}, S_{H,V})$ is trivial by Theorem 14.3 of [7], as $H \neq Q$ by hypothesis. $\square$

The problem now is to determine when it is possible to restrict the general case to the case where the subgroups indexing the simple functors are normal. To answer this question, we will introduce functors, called functors T, whose definition is similar to those of simple functors determined in the classification, except that they are not indexed by simple modules.

Definition 2.2. Let H be a subgroup of G and V a $k\overline{N}_G(H)$-module. The Mackey functor $T_{H,V}$ is the subfunctor of $M = (\mathrm{Inf}^{N_G(H)}_{\overline{N}_G(H)} FP_V){\uparrow}^G_{N_G(H)}$ that is generated by $T_{H,V}(H) = M(H) = V$.

These functors may be characterized in the following way:

Lemma 2.3. *Let V be a $k\overline{N}_G(H)$-module, where $H \leqslant G$. Let T_V be the subfunctor of FP_V, for the group $\overline{N}_G(H)$, defined by $T_V(J) = I^J_1(V)$. The functor $T_{H,V}$ is then isomorphic to $(\mathrm{Inf}^{N_G(H)}_{\overline{N}_G(H)} T_V){\uparrow}^G_{N_G(H)}$.*

Proof. The proof is similar to the one of Lemma 8.1 of [8]. $\square$

The functors $T_{H,V}$ may be defined using G-sets (with the definition of Dress).

Proposition 2.4. *Let V be a $k\overline{N}_G(H)$-module, where $H \leqslant G$, and let X be a G-set. Then,*

$$T_{H,V}(X) = I_1^{\overline{N}_G(H)}\big(\mathrm{Hom}_k\big(k\big(X^H\big), V\big)\big)$$

where $I_1^{\overline{N}_G(H)}$ is the relative trace map.

Proof. First suppose that $H = 1$. We can assume that X is a transitive G-set; or in other words $X = G/J$ and we shall show that

$$I_1^J(V) \cong I_1^G\big(\mathrm{Hom}_k(kG/J, V)\big).$$

Remark that $I_1^J(V) \subseteq V^J$ and $I_1^G(\mathrm{Hom}_k(kG/J, V)) \subseteq (\mathrm{Hom}_k(kG/J, V))^G$. Moreover, the Frobenius reciprocity theorem leads us to the following isomorphism:

$$\varphi : V^J \xrightarrow{\sim} \mathrm{Hom}_{kJ}(k, V) \xrightarrow{\sim} \mathrm{Hom}_{kG}(kG/J, V) \xrightarrow{\sim} \big(\mathrm{Hom}_k(kG/J, V)\big)^G.$$

In particular, if $x = \sum_{j\in J} jv \in I_1^J(V)$ with $v \in V$, then $\varphi(x)$ is the kG-linear map defined by $\varphi(x)(u) = \varphi(x)(1 \cdot J) = x$.
Next, let us show that $\varphi(x)$ belongs to $I_1^G(\mathrm{Hom}_k(kG/J, V)$. In order to do so, let us define the k-linear map $f : kG/J \to V$ by

$$f(gJ) = \begin{cases} x & \text{si } g \in J, \\ 0 & \text{sinon.} \end{cases}$$

Then,

$$I_1^G(f)(u) = \sum_{g\in G} g^{-1} f(gu) = \sum_{j\in J} \sum_{t\in[G/J]} j^{-1}t^{-1} f(tju) = \sum_{j\in J} j^{-1}\Bigg(\sum_{t\in[G/J]} t^{-1} f(tJ)\Bigg) = \sum_{j\in J} j^{-1}v = x.$$

Consequently, $\varphi(x) = I_1^G(f)$, and so $\varphi(I_1^J(V)) \subseteq I_1^G(\mathrm{Hom}_k(kG/J, V))$.

Moreover, if $\tilde{f} \in \mathrm{Hom}_k(kG/J, V)$, then

$$\begin{aligned} I_1^G\big(\tilde{f}(u)\big) &= \sum_{g\in G} g^{-1}\tilde{f}(gu) = \sum_{j\in J}\sum_{t\in[G/J]} j^{-1}t^{-1}\tilde{f}(tjJ) = \sum_{j\in J} j^{-1}\Bigg(\sum_{t\in[G/J]} t^{-1}\tilde{f}(tJ)\Bigg) \\ &= \varphi\Bigg(\sum_{t\in[G/J]} t^{-1}\tilde{f}(tJ)\Bigg). \end{aligned}$$

Thus, φ induces an isomorphism from $I_1^J(V)$ onto $I_1^G(\mathrm{Hom}_k(kG/J, V))$.

Using the previous result, we obtain, for every G-set X,

$$\begin{aligned} T_{H,V}(X) &= \mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)}\big(T_{1,V}\big(\mathrm{Res}^G_{N_G(H)}(X)\big)\big) = T_{1,V}\big(\big(\mathrm{Res}^G_{N_G(H)}(X)\big)^H\big) \\ &= I_1^{\overline{N}_G(H)}\big(\mathrm{Hom}_k\big(k\big(\big(\mathrm{Res}^G_{N_G(H)}(X)\big)^H\big), V\big)\big) \\ &= I_1^{\overline{N}_G(H)}\big(\mathrm{Hom}_k\big(k\big(X^H\big), V\big)\big). \qquad \square \end{aligned}$$

These functors T possess properties that entirely characterize them (in a similar way as simple Mackey functors).

Proposition 2.5. *Let M be a Mackey functor for G and H a minimal subgroup such that $M(H) \neq 0$. Let χ be the subconjugacy closure of H, that is all the subgroups of G which are conjugate to a subgroup of H. Then, $M \cong T_{H,V}$, where $V = M(H)$, if and only if the following conditions are satisfied*:

(a) $\mathrm{Im}(I_\chi) = M$, *where* $\mathrm{Im}(I_\chi)$ *is the subfunctor of M defined by*

$$\mathrm{Im}(I_\chi)(K) = \sum_{X \in \chi,\, X \leqslant K} \mathrm{Im}\big(I_X^K\big),$$

(b) $\mathrm{Ker}(R_\chi) = 0$, *where* $\mathrm{Ker}(R_\chi)$ *is the subfunctor of M defined by*

$$\mathrm{Ker}(R_\chi)(K) = \bigcap_{X \in \chi,\, X \leqslant K} \mathrm{Ker}\big(R_X^K\big).$$

Proof. If $M = T_{H,V}$, then H is by definition a minimal subgroup of M. Moreover, as M is generated by its value in H, the application $\sum_{g \in T_G(H,K)} I_{{}^gH}^K$ maps $M({}^gH)$ onto $M(K)$ (which is zero if H is not conjugate to a subgroup of K). Thus

$$\mathrm{Im}(I_\chi)(K) = \sum_{X \in \chi,\, X \leqslant K} \mathrm{Im}\big(I_X^K\big) = \sum_{g \in T_G(H,K)} \mathrm{Im}\big(I_{{}^gH}^K\big) = T_{H,V}(K)$$

and consequently $\mathrm{Im}(I_\chi) = T_{H,V}$. In a similar way,

$$\mathrm{Ker}(R_\chi)(K) = \bigcap_{X \in \chi,\, X \leqslant K} \mathrm{Ker}\big(R_X^K\big) = \bigcap_{g \in T_G(H,K)} \mathrm{Ker}\big(R_{{}^gH}^K\big).$$

Fix $g \in T_G(H,K)$. The application

$$R_{{}^gH}^K : T_{H,V}(K) = \bigoplus_{x \in [K \backslash T_G(H,K)/N_G(H)]} I_H^{N_{K^x}(H)}(V) \to T_{H,V}\big({}^gH\big) = \bigoplus_{y \in \{g\}} V$$

is given by $R_{{}^gH}^K(y)_g = R_{N_G(H) \cap {}^gH}^{N_G(H) \cap K}(y_g) = y_g$, for restrictions of a fixed point functor are inclusions. Hence, if $y \in \mathrm{Ker}(R_\chi)(K)$, then $y_g = 0$ for every $g \in T_G(H,K)$. Consequently, $\mathrm{Ker}(R_\chi) = 0$.

Conversely, assuming that M is a Mackey functor satisfying the above properties, H is then a minimal subgroup such that $M(H) \neq 0$ and Proposition 1.1 leads us to the following isomorphism (where $V = M(H)$):

$$\mathrm{Hom}_{\mu_k(G)}\big(M, \big(\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} FP_V\big)\uparrow_{N_G(H)}^G\big) \to \mathrm{Hom}_{k\overline{N}_G(H)}(V, V).$$

In particular, to id_V corresponds the application $\varphi = (\varphi(K))_K$ where $\varphi(K) = 0$ if $H \nleqslant_G K$ and, if $H \leqslant_G K$,

$$\begin{aligned} \varphi(K): \quad M(K) &\to \bigoplus_{g \in I_K} FP_V\big(\overline{N}_{K^g}(H)\big), \\ a &\mapsto \big(c_g R_{{}^gH}^K(a)\big)_{g \in I_K}, \end{aligned}$$

where $I_K = [K \backslash T_G(H,K)/N_G(H)]$. Consequently, if $a \in \mathrm{Ker}(\varphi(K))$ then a belongs to $\bigcap_{g \in T_G(H,K)} \mathrm{Ker}(R_{{}^gH}^K) = \mathrm{Ker}(R_\chi)(K)$, which is trivial; or in other words φ is injective. Hence, M may be identified with a subfunctor of $(\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} FP_V)\uparrow_{N_G(H)}^G$, and as $M = \mathrm{Im}(I_\chi)$, M is generated by its value in H, so $M \cong T_{H,M(H)}$. $\square$

Let us next remark that if H is a subgroup of G, we may associate to each $k\overline{N}_G(H)$-module V the Mackey functor $T_{H,V}$. Moreover, this application is functorial and preserves injections and surjections.

Proposition 2.6. *Let H be a subgroup of G. There exists a functor $\mathcal{T}_H$ from the category of $k\overline{N}_G(H)$-modules to the category of Mackey functors, defined by $\mathcal{T}_H(V) = T_{H,V}$. Moreover, $\mathcal{T}_H$ is a fully faithful functor that preserves injections and surjections.*

Proof. Let $f : V \to W$ be a homomorphism of $k\overline{N}_G(H)$-modules. The Mackey functors morphism $\mathcal{T}_H(f) : T_{H,V} \to T_{H,W}$ is defined in the following way: if $H \leqslant_G J$,

$$\mathcal{T}_H(f)(J): \quad T_{H,V}(J) = \bigoplus_{g\in I} I_H^{N_{J^g}(H)}(V) \to T_{H,W}(J) = \bigoplus_{g\in I} I_H^{N_{J^g}(H)}(W)$$
$$y = (y_g)_{g\in I} \mapsto \big(f(y_g)\big)_{g\in I}$$

where $I = [J\backslash T_G(H, J)/N_G(H)]$. The application $\mathcal{T}_H(f)$ is a Mackey functors morphism since f is a $k\overline{N}_G(H)$-linear map.

We shall show next that $\mathcal{T}_H$ is fully faithful. On the one hand, if $\mathcal{T}_H(f)$ is zero, then $f = \mathcal{T}_H(f)(H) = 0$. On the other hand, if $\psi : T_{H,V} \to T_{H,W}$ is a Mackey functors morphism, then $\psi = \mathcal{T}_H(\psi(H))$. Indeed, using Proposition 2.5, one only need to check that the evaluations at H of these two morphisms are the same; one have $\mathcal{T}_H(\psi(H))(H) = \psi(H)$, by definition of the functor $\mathcal{T}_H$. Finally, we verify that the functor $\mathcal{T}_H$ preserves injections and surjections using its definition. $\square$

These functors $T_{H,V}$ are a generalization of simple functors, as they are simple precisely when the module V is simple:

Proposition 2.7. *Let H be a subgroup of G and V a $k\overline{N}_G(H)$-module. The functor $T_{H,V}$ is simple if and only if the module V is simple.*

Proof. If V is simple, then the definition of the functor $T_{H,V}$ coincides with that of the simple functor $S_{H,V}$, and hence $T_{H,V}$ is simple. Conversely, if V possesses a proper non-zero submodule U, then using Proposition 2.6, we obtain that $T_{H,U}$ is a proper non-zero subfunctor of $T_{H,V}$. Consequently, $T_{H,V}$ is not simple. $\square$

These functors are also indecomposable precisely when the corresponding module is indecomposable:

Proposition 2.8. *Let $H \leqslant G$ and V a $k\overline{N}_G(H)$-module. The functor $T_{H,V}$ is indecomposable if and only if the module V is indecomposable. More precisely, $V = U \oplus W$ if and only if $T_{H,V} = T_{H,U} \oplus T_{H,W}$.*

Proof. Suppose that V is indecomposable. If $T_{H,V} = M_1 \oplus M_2$, then evaluating at H leads to $V = M_1(H) \oplus M_2(H)$, and using the hypothesis, we may assume that $V = M_1(H)$, without lost of generality. Moreover, the fact that the functor $T_{H,V}$ is generated by its value in H implies that $T_{H,V} = M_1$ and $M_2 = 0$ and consequently, $T_{H,V}$ is indecomposable. Conversely, suppose that $V = U \oplus W$ with U and W non-zero. Then, we obtain $T_{H,V} = T_{H,U} \oplus T_{H,W}$, using Proposition 2.6. In particular, the functor $T_{H,V}$ is not indecomposable. $\square$

As these functors T are not simple if V is not a simple $k\overline{N}_G(H)$-module, it is a natural question to compute their socle and their head:

Proposition 2.9. *Let H be a subgroup of G and V a $k\overline{N}_G(H)$-module whose socle is equal to $\mathrm{Soc}(V) = \bigoplus_i V_i$, where the V_i are simple $k\overline{N}_G(H)$-modules for every i, and whose head is equal to $\mathrm{Hd}(V) = \bigoplus_j W_j$, where the W_j are simple $k\overline{N}_G(H)$-modules. Then, $\mathrm{Soc}(T_{H,V}) = \bigoplus_i S_{H,V_i}$ and $\mathrm{Hd}(T_{H,V}) = \bigoplus_j S_{H,W_j}$.*

Proof. As $\mathrm{Soc}(V)$ is a $k\overline{N}_G(H)$-submodule of V, the subfunctor M of $T_{H,V}$ generated by $M(H) = \mathrm{Soc}(V)$ is a subfunctor of $(\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)} FP_V)\uparrow_{N_G(H)}^G$. Therefore, $M \cong T_{H,\mathrm{Soc}(V)}$ by definition of the functors T. As $T_{H,\mathrm{Soc}(V)} \cong \bigoplus_i S_{H,V_i}$ (see Proposition 2.8), it is a semisimple subfunctor of $T_{H,V}$. Let us prove that it is the socle of $T_{H,V}$. Let $S \cong S_{K,U}$ be a simple subfunctor of $T_{H,V}$. As $S(K) = U \neq 0$, we must have $H \leqslant_G K$. Let $g \in G$ be such that ${}^gH \leqslant K$. As restrictions are injective, $R^K_{{}^gH}(S(K)) \neq 0$, hence $S({}^gH) \neq 0$, which implies $H =_G K$ by minimality of K. Thus, $S \cong S_{H,U}$, where U is a simple submodule of V; or in other words, there exists an index i such that $U \cong V_i$. Therefore, S is a subfunctor of $T_{H,\mathrm{Soc}(V)}$.

Let us now focus on the head of $T_{H,V}$. As $\mathrm{Hd}(V)$ is a quotient $k\overline{N}_G(H)$-module of V, there exists a surjective homomorphism $\varphi : V \to \mathrm{Hd}(V)$. Let $\pi_j : \mathrm{Hd}(V) \to W_j$ be the canonical projection. This implies that $\varphi_j = \pi_j \circ \varphi$ is non-zero for every index j. By Proposition 1.1, for every j there exists a non-zero homomorphism μ_j from $T_{H,V}$ to $M_j = (\mathrm{Inf}_{\overline{N}_G(K)}^{N_G(K)} FP_{W_j})\uparrow_{N_G(K)}^G$, whose image is contained in the socle of M_j which equals S_{H,W_j} by Lemma 8.1 of [8]. Hence, the image of the application

$$\mu = \bigoplus_j \mu_j : T_{H,V} \to \bigoplus_j M_j \cong \big(\mathrm{Inf}_{\overline{N}_G(K)}^{N_G(K)} FP_{\mathrm{Hd}(V)}\big)\uparrow_{N_G(K)}^G$$

is equal to $\bigoplus_j S_{H,W_j}$ which is isomorphic to $T_{H,\mathrm{Hd}(V)}$, by Proposition 2.8. Thus, $T_{H,\mathrm{Hd}(V)}$ is a semisimple quotient of the functor $T_{H,V}$. Let us show that it is the head of $T_{H,V}$. Assuming that $S_{K,U}$ is a simple quotient of $T_{H,V}$, there exists a non-zero homomorphism $T_{H,V} \to S_{K,U}$. In particular, there exists a non-zero application $T_{H,V} \to M = (\mathrm{Inf}_{\overline{N}_G(K)}^{N_G(K)} FP_U)\uparrow_{N_G(K)}^G$ whose image is contained in the socle of M which is precisely equal to $S_{K,U}$ by Lemma 8.1 of [8]. By Proposition 1.1 this implies the existence of a non-zero morphism $\alpha : T^+_{H,V}(1) = T^+_{H,V}(K/K) \to U$. Yet

$$T^+_{H,V}(1) = T_{H,V}(K)\Big/\Bigg(\sum_{J \leqslant K,\, J \ngeqslant K} I^K_J(V)\Bigg) = \begin{cases} V & \text{if } K =_G H, \\ 0 & \text{otherwise,} \end{cases}$$

for if $T_{H,V}(K) \neq 0$, then $H \leqslant_G K$ and if $H <_G K$, then $T_{H,V}(K)$ is induced from H; hence, the above quotient is zero. Consequently, $K =_G H$ which leads to the existence of a non-zero homomorphism φ of $k\overline{N}_G(H)$-modules from V to U; or in other words, there exists an index j such that $U \cong W_j$. Thus, S is a quotient of $T_{H,\mathrm{Hd}(V)}$. $\square$

Functors T are more complicated than simple functors, but they have the property of being stable by restriction, which will be very useful for restricting extension groups between functors T to those between functors T indexed by normal subgroups.

Proposition 2.10. *Let H and L be subgroups of G, and let V be a $k\overline{N}_G(H)$-module. Then,*

$$T^G_{H,V}\downarrow^G_L = \bigoplus_{g \in I} T^L_{{}^gH, c_g(V)},$$

where $I = [L \backslash T_G(H,L)/N_G(H)]$.

Proof. By Lemma 2.3, $T_{H,V} = (\mathrm{Inf}_{\overline{N}_G(H)}^{N_G(H)}\ T_{1,V})\uparrow_{N_G(H)}^G$ and hence, using Mackey formula,

$$T^G_{H,V}\downarrow^G_L = T^{N_G(H)}_{H,V}\uparrow^G_{N_G(H)}\downarrow^G_L = \bigoplus_{g \in I}\big(c_g\big(T^{N_G(H)}_{H,V}\downarrow^{N_G(H)}_{N_{L^g}(H)}\big)\big)\uparrow^L_{N_L({}^gH)}.$$

Moreover, if $K \leqslant N_{L^g}(H)$, then

$$T_{H,V}^{N_G(H)}\downarrow_{N_{L^g}(H)}^{N_G(H)}(K) = T_{H,V}^{N_G(H)}(K) = T_{H,V}^{N_{L^g}(H)}(K)$$

meaning that $T_{H,V}^{N_G(H)}\downarrow_{N_{L^g}(H)}^{N_G(H)} = T_{H,V}^{N_{L^g}(H)}$. Note by the way that this property is not verified for simple Mackey functors, because usually the module V is not simple as $\overline{N}_{L^g}(H)$-module. Similarly if $K \leqslant N_L({}^gH)$, then

$$c_g\big(T_{H,V}^{N_{L^g}(H)}\big)(K) = T_{H,V}^{N_{L^g}(H)}\big(K^g\big) = I_1^{K^g/H}(V) = I_1^{K/{}^gH}\big(c_g(V)\big) = T_{{}^gH,c_g(V)}^{N_L({}^gH)}(K),$$

or in other words, $c_g(T_{H,V}^{N_{L^g}(H)}) = T_{{}^gH,c_g(V)}^{N_L({}^gH)}$. Consequently,

$$T_{H,V}^G\downarrow_L^G = \bigoplus_{g\in I} T_{{}^gH,c_g(V)}^{N_L({}^gH)}\uparrow_{N_L({}^gH)}^L = \bigoplus_{g\in I} T_{{}^gH,c_g(V)}^L,$$

where the last equality comes from the definition of the functor $T_{{}^gH,c_g(V)}^L$, for $g \in I$. $\square$

By the way, this Proposition enables us to determine precisely when the restriction of a simple Mackey functor is semisimple:

Proposition 2.11. *Let H and L be subgroups of G, and let V be a simple $k\overline{N}_G(H)$-module. If $H \nleqslant_G L$, then $S_{H,V}^G\downarrow_L^G = 0$. Otherwise, the Mackey functor $S_{H,V}^G\downarrow_L^G$ is semisimple if and only if the modules $\mathrm{Res}_{\overline{N}_L({}^gH)}^{\overline{N}_G({}^gH)}(c_g(V))$, that is the restriction of the conjugate of the module V, are semisimple for every g in $I = [L\backslash T_G(H,L)/N_G(H)]$.*

Proof. If $H \nleqslant_G L$, then $S_{H,V}^G\downarrow_L^G = 0$, by Proposition 2.10. Therefore, let us assume that $H \leqslant_G L$. First suppose that $\mathrm{Res}_{\overline{N}_L({}^gH)}^{\overline{N}_G({}^gH)}(c_g(V)) \cong \bigoplus_i V_{g,i}$ where the modules $V_{g,i}$ are simple for every $g \in I$. Thanks to Proposition 2.10, we have $S_{H,V}^G\downarrow_L^G = \bigoplus_{g\in I} T_{{}^gH,c_g(V)}^L$. Using Proposition 2.8, we obtain $S_{H,V}^G\downarrow_L^G \cong \bigoplus_{g\in I}\bigoplus_i S_{{}^gH,V_{g,i}}^L$; or in other words, the functor $S_{H,V}^G\downarrow_L^G$ is semisimple.

Conversely, let us assume that $S_{H,V}^G\downarrow_L^G = \bigoplus_i S_{H_i,V_i}$ is a direct sum of simple Mackey functors. In particular, the V_i are simple $k\overline{N}_L(H_i)$-modules. By Proposition 2.10, $S_{H,V}^G\downarrow_L^G = \bigoplus_{g\in I} T_{{}^gH,c_g(V)}^L$. By the Krull–Schmidt theorem, it follows that for every $g \in I$, there exist indexes i_g such that $T_{{}^gH,c_g(V)}^L = \bigoplus_{i_g} S_{H_{i_g},V_{i_g}}$. In addition, the summands of the socle of the functors $T_{{}^gH,c_g(V)}^L$ are only simple Mackey functors indexed by the subgroup H (see Proposition 2.9). Hence, $H_{i_g} = {}^gH$ for every index i_g. By evaluating these expressions at ${}^gH$, we deduce that $c_g(V) = \bigoplus_{i_g} V_{i_g}$ for every $g \in I$ and thus the modules $c_g(V)$ are semisimple $k\overline{N}_L({}^gH)$-modules. $\square$

Finally, let us point out that these functors T are also stable by induction.

Proposition 2.12. *Let $H \leqslant J \leqslant G$ and V a $k\overline{N}_J(H)$-module. Then,*

$$T_{H,V}^J\uparrow_J^G = T_{H,\mathrm{Ind}_{\overline{N}_J(H)}^{\overline{N}_G(H)}(V)}^G$$

where $\mathrm{Ind}_{\overline{N}_J(H)}^{\overline{N}_G(H)}(V)$ is the induced of the module V from $\overline{N}_J(H)$ to $\overline{N}_G(H)$.

Proof. First of all, let us remark that for every subgroup L of a group M and for every kM-module A and every kL-module B, we have

$$I_1^M\big(\mathrm{Hom}_k\big(A, \mathrm{Ind}_L^M(B)\big)\big) \cong I_1^L\big(\mathrm{Hom}_k(A, B)\big)$$

where I_1^M and I_1^L are the relative trace map from 1 to M and L respectively (for details, see [5, Proposition 2.3.13]).

We will use the definition of the functors T using G-set (see Proposition 2.4). Let X be a G-set. Using the previous result we obtain

$$\begin{aligned} T_{H,V}\uparrow_J^G(X) &= T_{H,V}^J\big(\mathrm{Res}_J^G(X)\big) = I_1^{\overline{N}_J(H)}\big(\mathrm{Hom}_k\big(k\big(X^H\big), V\big)\big) \\ &= I_1^{\overline{N}_G(H)}\big(\mathrm{Hom}_k\big(k\big(X^H\big), \mathrm{Ind}_{\overline{N}_J(H)}^{\overline{N}_G(H)}(V)\big)\big) \\ &= T^G_{H,\mathrm{Ind}_{\overline{N}_J(H)}^{\overline{N}_G(H)}(V)}(X). \qquad \square \end{aligned}$$

We are now able to prove that the calculation of extension groups between functors T restricts itself to the case where these functors T are indexed by normal subgroups.

Proposition 2.13. *Let H and Q be subgroups of G, V a $k\overline{N}_G(H)$-module and W a $k\overline{N}_G(Q)$-module. Let $J = [N_G(Q)\backslash T_G(H, N_G(Q))/N_G(H)]$, $N = N_G(Q)$ and for every $g \in J$ let $M_g = N_G({}^gH) \cap N$. Then*

$$\mathrm{Ext}^1_{\mu_k(G)}\big(T^G_{Q,W}, T^G_{H,V}\big) = \bigoplus_{\substack{g\in J \text{ such that} \\ Q\leqslant N_G({}^gH)}} \mathrm{Ext}^1_{\mu_k(M_g)}\big(T^{M_g}_{Q,W}, T^{M_g}_{{}^gH,c_g(V)}\big)$$

or in other words, the calculation of extension groups between functors T restricts itself to the case where these functors are indexed by normal subgroups.

Proof. Using the properties of adjunctions between restriction and induction (see Proposition 4.2 of [8]) and Proposition 2.10, we obtain:

$$\begin{aligned} \mathrm{Ext}^1_{\mu_k(G)}\big(T^G_{Q,W}, T^G_{H,V}\big) &\cong \mathrm{Ext}^1_{\mu_k(G)}\big(T^N_{Q,W}\uparrow_N^G, T^G_{H,V}\big) \cong \mathrm{Ext}^1_{\mu_k(N)}\big(T^N_{Q,W}, T^G_{H,V}\downarrow_N^G\big) \\ &\cong \bigoplus_{g\in J} \mathrm{Ext}^1_{\mu_k(N)}\big(T^N_{Q,W}, T^N_{{}^gH,c_g(V)}\big) \\ &\cong \bigoplus_{g\in J} \mathrm{Ext}^1_{\mu_k(N)}\big(T^N_{Q,W}, T^{M_g}_{{}^gH,c_g(V)}\uparrow^N_{M_g}\big) \\ &\cong \bigoplus_{g\in J} \mathrm{Ext}^1_{\mu_k(M_g)}\big(T^N_{Q,W}\downarrow^N_{M_g}, T^{M_g}_{{}^gH,c_g(V)}\big) \\ &\cong \bigoplus_{\substack{g\in J \text{ such that} \\ Q\leqslant N_G({}^gH)}} \mathrm{Ext}^1_{\mu_k(M_g)}\big(T^{M_g}_{Q,W}, T^{M_g}_{{}^gH,c_g(V)}\big), \end{aligned}$$

where the last isomorphism comes from the fact that $T^N_{Q,W}\downarrow^N_{M_g} = \bigoplus_{h\in I} T^{M_g}_{{}^hQ,c_h(W)}$ with $I = [M_g\backslash T_N(Q, M_g)/N_N(Q)]$. Hence, $I = \{1\}$ if $Q \leqslant N_G({}^gH)$ and $I = \emptyset$ otherwise. $\square$

We are now able to characterize the situations where determining extension groups of degree 1 between simple Mackey functors restricts itself to the case where simple functors are indexed by normal subgroups. Moreover, in these situations, we can even compute these groups explicitly.

Proposition 2.14. *Let H and Q be subgroups of a group G, V a simple $k\overline{N}_G(H)$-module and W a simple $k\overline{N}_G(Q)$-module. Let $I = [N_G(Q)\backslash T_G(H, N_G(Q))/N_G(H)]$, $N = N_G(Q)$ and for every $g \in I$ let $M_g = N_G({}^gH) \cap N$. Furthermore, assume that the two following conditions are satisfied*:

(i) *for every $g \in I$, $\mathrm{Res}^{\overline{N}_G({}^gH)}_{\overline{N}_N({}^gH)}(c_g(V)) = \bigoplus_i V_{g,i}$, where the $V_{g,i}$ are simple $k\overline{N}_N({}^gH)$-modules,*
(ii) *for every $g \in I$, $\mathrm{Res}^{\overline{N}_G(Q)}_{\overline{N}_{M_g}(Q)}(W) = \bigoplus_j W_{g,j}$, where the $W_{g,j}$ are simple $k\overline{N}_{M_g}(Q)$-modules.*

Then, the calculation of extension groups between simple Mackey functors restricts itself to the case where these functors are indexed by normal subgroups. More precisely,

$$\mathrm{Ext}^1_{\mu_k(G)}\big(S^G_{Q,W}, S^G_{H,V}\big) = \bigoplus_{\substack{g\in I \text{ such that}\\ Q\leqslant N_G({}^gH)}} \bigoplus_{i,j} \mathrm{Ext}^1_{\mu_k(M_g)}\big(S^{M_g}_{Q,W_{g,j}}, S^{M_g}_{{}^gH,V_{g,i}}\big)$$
$$\cong \begin{cases} \bigoplus_{g\in I_1}\bigoplus_{i,j\in I_2} k & \text{if there exists } x\in G \text{ such that } {}^xH < Q \text{ and } [Q:{}^xH]=p,\\ \bigoplus_{g\in J_1}\bigoplus_{i,j\in J_2} k & \text{if there exists } x\in G \text{ such that } Q < {}^xH \text{ and } [{}^xH:Q]=p,\\ 0 & \text{otherwise,}\end{cases}$$

where I_1 (respectively J_1) is the set of elements $g \in I$ such that ${}^gH < Q \leqslant N_G({}^gH)$ (respectively such that $Q < {}^gH$) and where I_2 (respectively J_2) is the set of indexes i and j such that $V_{g,i} \cong W_{g,j}$ and $FP_{V_{g,i}}(Q/{}^gH) = V_{g,i}$ (respectively such that $FP_{W_{g,j}}({}^gH/Q) = W_{g,j}$).

Proof. It follows directly from Propositions 2.1 and 2.13. □

This result gives us the values of the extension groups for simple Mackey functors for a p-group, or more generally for simple Mackey functors indexed by the trivial module:

Theorem 2.15. *Let Q and H be non-conjugate subgroups of a group G. The k-vector space $\mathrm{Ext}(S_{Q,k}, S_{H,k})$ is then equal to*

$$\begin{cases} \bigoplus_{g\in J} k & \text{if there exists } x\in G \text{ such that } {}^xH < Q \text{ and } [Q:{}^xH]=p,\\ \bigoplus_{g\in J'} k & \text{if there exists } x\in G \text{ such that } {}^xQ < H \text{ and } [H:{}^xQ]=p,\\ 0 & \text{otherwise,}\end{cases}$$

where J (respectively J') is the set of elements $g \in [N_G(Q)\backslash T_G(H, N_G(Q))/N_G(H)]$ such that ${}^gH \trianglelefteq Q$ (respectively such that $Q \trianglelefteq {}^gH$).

Proof. As restriction of the trivial module k to any subgroup of G is again the trivial module, we can apply Proposition 2.14 to obtain the result. □

We can finally deal with the case where the group G possesses a normal Sylow p-subgroup.

Theorem 2.16. *Let G be a group possessing a normal Sylow p-subgroup P. Let Q and H be non-conjugate subgroups of G, let V be a simple $k\overline{N}_G(H)$-module and let W be a simple $k\overline{N}_G(Q)$-module. Let $I = [N_G(Q)\backslash T_G(H, N_G(Q))/N_G(H)]$, and for every $g \in I$, let $N = N_G(Q)$ and $M_g = N_G({}^gH) \cap N$. For every $g \in I$, the modules $\mathrm{Res}^{\overline{N}_G({}^gH)}_{\overline{N}_N({}^gH)}(c_g(V))$ (respectively $\mathrm{Res}^{\overline{N}_G(Q)}_{\overline{N}_{M_g}(Q)}(c_g(W))$) decompose as a direct sum of simple submodules $V_{g,i}$ (respectively $W_{g,j}$), as in Proposition 2.14. The k-vector space $\mathrm{Ext}(S_{Q,W}, S_{H,V})$ is then equal to*

$$\begin{cases} \bigoplus_{g\in I_1}\bigoplus_{i,j\in I_2} k & \textit{if there exists } x\in G \textit{ such that } {}^xH < Q \textit{ and } [Q:{}^xH]=p, \\ \bigoplus_{g\in J_1}\bigoplus_{i,j\in J_2} k & \textit{if there exists } x\in G \textit{ such that } Q < {}^xH \textit{ and } [{}^xH:Q]=p, \\ 0 & \textit{otherwise,} \end{cases}$$

where I_1 (respectively J_1) is the set of elements $g \in I$ such that ${}^gH \trianglelefteq Q$ (respectively such that $Q \trianglelefteq {}^gH$) and where I_2 (respectively J_2) is the set of indexes i and j such that $V_{g,i} \cong W_{g,j}$ and $FP_{V_{g,i}}(Q/{}^gH) = V_{g,i}$ (respectively such that $FP_{W_{g,j}}({}^gH/Q) = W_{g,j}$).

Proof. By Proposition 2.14, one needs only to check that the modules $\mathrm{Res}^{\overline{N}_G({}^gH)}_{\overline{N}_N({}^gH)}(c_g(V))$ and $\mathrm{Res}^{\overline{N}_G(Q)}_{\overline{N}_{M_g}(Q)}(c_g(W))$ decompose as a direct sum of simple submodules, for every $g \in I$. This comes from two facts. Firstly, for any simple kG-module U, where G is a group which has a normal Sylow p-subgroup, its restriction to any subgroup of G is semisimple. Secondly, the groups $N_G({}^gH)/{}^gH$ and $N_G(Q)/Q$ also possess a normal Sylow p-subgroup. Hence, we can apply the previous result to the modules $c_g(V)$ and $c_g(W)$. □

We will conclude this section by exhibiting certain cases where the extension groups between functors T are trivial. To begin with, we have the following result:

Proposition 2.17. *Let H and Q be subgroups of G, let V be a $k\overline{N}_G(H)$-module and let W be a $k\overline{N}_G(Q)$-module. The group $\mathrm{Ext}(T_{Q,W}, T_{H,V})$ is trivial unless $H \leqslant_G Q$ or $Q \leqslant_G H$.*

Proof. Assume that $H \nleqslant_G Q$ and that $Q \nleqslant_G H$. Let $\mathcal{E}: 0 \to T_{H,V} \xrightarrow{i} M \xrightarrow{p} T_{Q,W} \to 0$ be an extension of Mackey functors. Let $U = \mathrm{Soc}(V) = \bigoplus_i V_i$, where the V_i are simple modules, and let $P_U = \bigoplus_i P_{V_i}$ be the projective cover of the module U. Then, $M(J) = 0$ if $J < H$. Indeed, if $J < H$, then $T_{H,V}(J) = 0$ and $T_{Q,W}(J) = 0$, otherwise $Q \leqslant_G J < H$ which would contradict our hypothesis. Thus, by Proposition 1.1, there is an isomorphism

$$\mathrm{Hom}\big(M, \big(\mathrm{Inf}^{N_G(H)}_{\overline{N}_G(H)} FP_{P_U}\big)\uparrow^G_{N_G(H)}\big) \cong \mathrm{Hom}\big(M(H), P_U\big)$$

given by evaluation at H.

The application $i(H)|_{V_i}: V_i \to M(H)$ is injective. Since P_U is an injective module, there exist applications $\varphi^i_H: M(H) \to P_{V_i}$ such that $\varphi^i_H \circ i(H) = f_i$, where f_i is the inclusion of V_i in P_{V_i}. Let $N = (\mathrm{Inf}^{N_G(H)}_{\overline{N}_G(H)} FP_{P_U})\uparrow^G_{N_G(H)}$ and let $\varphi: M \to N$ be the application corresponding to $\varphi_H = \bigoplus_i \varphi^i_H: M(H) \to P_U$ via the previous bijection. We shall show the next two results:

(i) the application $p|_{\mathrm{Ker}(\varphi)}: \mathrm{Ker}(\varphi) \to T_{Q,W}$ is surjective,
(ii) $\mathrm{Ker}(\varphi) \cap i(T_{H,V}) = 0$.

Let us begin with (i). As $H \nleqslant_G Q$, the module $T_{H,V}(Q)$ is zero and hence, the application $p(Q): M(Q) \to T_{Q,W}(Q) = W$ is an isomorphism. Moreover, $M(Q)/\mathrm{Ker}(\varphi)(Q)$ embeds in $N(Q)$ which is equal to zero since Q does not contain any conjugate of H. Thus, $M(Q) = \mathrm{Ker}(\varphi)(Q)$. Consequently, for $J \leqslant G$,

$$\begin{aligned} T_{Q,W}(J) &= \sum_{g\in T_G(Q,J)} I^J_{{}^gQ}(W) = \sum_{g\in T_G(Q,J)} I^J_{{}^gQ}\big(p_{{}^gQ}\big(M\big({}^gQ\big)\big)\big) \\ &= \sum_{g\in T_G(Q,J)} p_J\big(I^J_{{}^gQ}\big(M\big({}^gQ\big)\big)\big) = p_J\Bigg(\sum_{g\in T_G(Q,J)} I^J_{{}^gQ}\big(M\big({}^gQ\big)\big)\Bigg) \end{aligned}$$

and since $\sum I^J_{{}^gQ}(M({}^gQ)) \subset \mathrm{Ker}(\varphi)(J)$, the result (i) is proved.

For the second result, one only need to show that φ is injective on the socle of $i(T_{H,V})$. Given that the socle of $T_{H,V}$ equals $\bigoplus_i S_{H,V_i}$, by Proposition 2.9, we deduce that, for every i, $\varphi|_{i(S_{H,V_i})} \neq 0$. Indeed,

$$\varphi_H^i i(H)\big(S_{H,V_i}(H)\big) = \varphi_H^i i(H)(V_i) = f_i(V_i)$$

which is non-zero due to f_i being the canonical inclusion of V_i in P_{V_i} and consequently, $\varphi_H|_{i(H)(S_{H,V_i}(H))} = \bigoplus_i \varphi_H^i|_{i(H)(S_{H,V_i}(H))} \neq 0$. Therefore, for every i, the application φ is non-zero on $i(S_{H,V_i})$, and so injective by simplicity of S_{H,V_i}, which proves point (ii).

Hence, the application $\tilde{p} = p|_{\mathrm{Ker}(\varphi)}$ is an isomorphism from $\mathrm{Ker}(\varphi)$ onto $T_{Q,W}$, by the two previous results. In particular, $0 \to T_{H,V} \xrightarrow{i} M \xrightarrow{p} T_{Q,W} \to 0$ is an exact sequence which is split via the application $\tilde{p}^{-1}$ or in other words, the extension $\mathcal{E}$ is trivial. $\square$

Using blocks of Mackey functors, we may refine the previous result.

Proposition 2.18. *Let H and Q be subgroups of G, V a $k\overline{N}_G(H)$-module and W a $k\overline{N}_G(Q)$-module. The group* $\mathrm{Ext}(T_{Q,W}, T_{H,V})$ *is trivial, unless the two following conditions are satisfied*:

(i) *There exists $g \in G$ such that ${}^gH \leqslant Q$ (respectively ${}^gQ \leqslant H$) and that $[Q : {}^gH]$ (respectively $[H : {}^gQ]$) is a power of p.*
(ii) *There exist direct summands W_0 and V_0 of W and V, respectively, such that $D^G = E^G$, where D^G is the block of kG corresponding to the block D of $kN_G(Q)$ to which W_0 belongs, and where E^G is the block of kG corresponding to the block E of $kN_G(H)$ to which V_0 belongs.*

Proof. First of all, thanks to Theorem 2.17, we can suppose that either $H \leqslant_G Q$ or $Q \leqslant_G H$. Using Proposition 2.8, we may also assume that the modules V and W are indecomposable. Consequently, if the group $\mathrm{Ext}(T_{Q,W}, T_{H,V})$ is non-trivial, then the functors $T_{Q,W}$ and $T_{H,V}$ have to belong to the same block of $\mathrm{Mack}_k(G)$. In particular, there exists a p-perfect subgroup J of G such that $T_{Q,W}$ and $T_{H,V}$ belong to $\mathrm{Mack}_k(G, J)$. As the simple subfunctors of $T_{H,V}$ (respectively $T_{Q,W}$) are indexed by H (respectively Q), it implies that $J = O^p(H) = O^p(Q)$. Hence, if ${}^gH \leqslant Q$, then $[Q : {}^gH]$ divides $[Q : O^p(Q)]$, and is therefore a power of p. We obtain a similar result if ${}^gQ \leqslant H$. The second condition comes from the description of blocks of Mackey functors given in Theorem 17.1 of [7]. $\square$

Let us finally point out that there can exist non-trivial extensions between functors T indexed by subgroups H and K with $H < K$ and $[K : H] > p$. In other words, the result on extensions between simple Mackey functors indexed by normal subgroups (see Theorem 2.1) does not generalize to the case of extensions between functors T.

Example. Consider the functor $T = T_{1,kP}$, where $P = C_{p^2} = \langle g \rangle$ is the cyclic group of order p^2. The module kP has a basis $\{v_1, \ldots, v_{p^2}\}$ such that the action of g is given by $gv_i = v_i + v_{i+1}$ for every $i = 1, \ldots, p^2 - 1$ and by $gv_{p^2} = v_{p^2}$. If H is equal to C_p or P, then the definition of the functor T leads to $T_{1,kP}(1) = kP$ and $T_{1,kP}(H) = I_1^H(kP)$. More precisely, we calculate that

$$I_1^{C_p}(v_i) = \begin{cases} v_{p^2-p+i} & \text{if } i \leqslant p, \\ 0 & \text{otherwise} \end{cases}$$

and

$$I_1^{P}(v_i) = \begin{cases} v_{p^2} & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $T_{1,kP}(P) = kv_{p^2} \cong k$ and $T_{1,kP}(C_p)$ possesses the k-basis $\{v_{p^2-p+1}, v_{p^2-p+2}, \ldots, v_{p^2}\}$. To fully describe the functor T, it is still necessary to mention that restrictions are inclusions and that the conjugation by an element x is the multiplication by x.

Let us now define the subfunctor N of T by $N(1)=\mathrm{vect}_k(v_2,\dots,v_{p^2})$, $N(C_p)=\mathrm{vect}_k(v_{p^2-p+2},\dots,v_{p^2})$ and $N(P)=kv_{p^2}$. The functor N has a maximal subfunctor L, isomorphic to $T_{1,\mathrm{Rad}(kP)}$ where $\mathrm{Rad}(kP)=\mathrm{vect}_k(v_2,\dots,v_{p^2})$. Indeed, if we set $L(P)=0$ and $L(J)=N(J)$ for $J<P$, one can verify that L is a subfunctor and that $N/L\cong S_{P,k}$. In addition, N has no subfunctor isomorphic to $S_{P,k}$. Consequently, there exists a non-split extension $0\to T_{1,\mathrm{Rad}(kP)}\to N\to S_{P,k}\to 0$ and thus, $\mathrm{Ext}(S_{P,k},T_{1,\mathrm{Rad}(kP)})\neq 0$ and $[P:1]=p^2$.

3. Extensions between simple Mackey functors indexed by the same subgroup

Let us now focus on the case of extension groups between simple Mackey functors indexed by the same subgroup. First of all, we can restrict ourselves to the case where this subgroup is normal. Indeed, by Proposition 2.13,

$$\mathrm{Ext}^1_{\mu_k(G)}\big(T^G_{H,V},T^G_{H,W}\big)=\bigoplus_{\substack{g\in J\text{ such that}\\ H\leqslant N_G({}^gH)}}\mathrm{Ext}^1_{\mu_k(M_g)}\big(T^{M_g}_{H,V},T^{M_g}_{{}^gH,c_g(W)}\big),$$

where $J=[N_G(H)\backslash T_G(H,N_G(H))/N_G(H)]$ and where the subgroup M_g is equal to $N_G({}^gH)\cap N_G(H)$, for every $g\in J$.

Moreover, by Theorem 2.17, the group $\mathrm{Ext}^1_{\mu_k(M_g)}(T^{M_g}_{H,V},T^{M_g}_{{}^gH,c_g(W)})$ is trivial unless $H\leqslant_{M_g}{}^gH$ or ${}^gH\leqslant_{M_g}H$. Nevertheless, as H is a normal subgroup of M_g, the group above is non-zero only if $g\in N_G(H)$. Consequently,

$$\mathrm{Ext}^1_{\mu_k(G)}\big(T^G_{H,V},T^G_{H,W}\big)=\mathrm{Ext}^1_{\mu_k(N_G(H))}\big(T^{N_G(H)}_{H,V},T^{N_G(H)}_{H,W}\big).\tag{3.0.1}$$

One possible approach for studying the group $\mathrm{Ext}(T_{H,V},T_{H,W})$ is to consider the morphism of evaluation at H:

$$\eta_H:\mathrm{Ext}^1_{\mu_k(G)}(T_{H,V},T_{H,W})\to\mathrm{Ext}^1_{k\overline{N}_G(H)}(V,W).$$

The first important property of this morphism is the following:

Proposition 3.1. *The morphism η_H is injective.*

Proof. As we have just seen, we may suppose that H is a normal subgroup of G. Let $\mathcal{E}:0\to T_{H,W}\xrightarrow{i}N\xrightarrow{p}T_{H,V}\to 0$ be an extension of Mackey functors such that the exact sequence of modules $0\to W\xrightarrow{i(H)}N(H)\xrightarrow{p(H)}V\to 0$ is split. Then, there exists $\sigma:N(H)\to W$ such that $\sigma i(H)=\mathrm{id}$. Let us prove that the extension $\mathcal{E}$ is trivial.

Thanks to Proposition 1.1, using the fact that $N(J)=0$ if $H\nleqslant_G J$, there is an isomorphism between $\mathrm{Hom}_{kG/H}(N(H),W)$ and $\mathrm{Hom}_{\mu_k(G)}(N,\mathrm{Inf}^G_{G/H}FP_W)$. In particular, to σ corresponds the morphism of Mackey functors $\tilde\sigma=(\tilde\sigma(K))_K$ which is zero if H is not a subgroup of K and which is given by

$$\begin{aligned}\tilde\sigma(K):\quad N(K)&\to FP_W(K/H),\\ a&\mapsto\sigma R^K_H(a)\end{aligned}$$

if $H\leqslant K$. In particular, if $H=K$, then $\tilde\sigma(H)=\sigma$. We need to prove that the image of $\tilde\sigma$ is contained in $T_{H,W}$ and that $\tilde\sigma i=\mathrm{id}$.

For the first point, if we keep in mind that $T_{H,W}$ is the subfunctor of $M=\mathrm{Inf}^G_{G/H}FP_W$, generated by $T_{H,W}(H)=M(H)=W$, all we need to show is that the image of $\tilde\sigma(K)$ is induced from $\tilde\sigma(H)$, for every subgroup K. Let K be such that $H \leqslant K \leqslant G$. As i and p are morphisms of Mackey functors, the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & T_{H,W}(H) \arrow[r, "i_H"] \arrow[d, "\alpha_1"] & N(H) \arrow[r, "p_H"] \arrow[d, "\beta"] & T_{H,V}(H) \arrow[r] \arrow[d, "\alpha_2"] & 0 \\ 0 \arrow[r] & T_{H,W}(K) \arrow[r, "i_K"] & N(K) \arrow[r, "p_K"] & T_{H,V}(K) \arrow[r] & 0 \end{tikzcd}$$

where α_1, α_2 and β equal I_H^K. As $T_{H,W}(K) = I_H^K(W)$ and $T_{H,W}(H) = W$, α_1 is surjective and, similarly, α_2 is surjective. Thanks to the five lemma [4], β is surjective. For any given $a \in N(K)$, there exists a $b \in N(H)$ such that $a = I_H^K(b)$. Hence, $\tilde{\sigma}(K)(a) = \tilde{\sigma}(K)I_H^K(b) = I_H^K\tilde{\sigma}(H)(b)$ and thus $\tilde{\sigma}(K)(N(K))$ is induced from $\tilde{\sigma}(H)(N(H))$.

All that remains to be proven is that $\tilde{\sigma}i = \mathrm{id}$. In order to do so, let us point out that the application of evaluation at H, $\mu_H : \mathrm{Hom}(T_{H,W}, T_{H,W}) \to \mathrm{Hom}(W, W)$, is injective. Indeed, let $\varphi : T_{H,W} \to T_{H,W}$ be such that $\mu_H(\varphi) = 0$, or in other words $\varphi(H) = 0$, and let $K \leqslant G$. If K does not contain H, then $T_{H,W}(K) = 0$, so $\varphi(K) = 0$; and otherwise

$$\varphi(K)\big(T_{H,W}(K)\big) = \varphi(K)\big(I_H^K(W)\big) = I_H^K\varphi(H)(W) = I_H^K\varphi(H)\big(T_{H,W}(H)\big) = 0.$$

Consequently, as $\mu_H(\tilde{\sigma}i) = \tilde{\sigma}(H)i(H) = \sigma i(H) = \mathrm{id}_H = \mu_H(\mathrm{id}_{T_{H,W}})$, we conclude that $\tilde{\sigma}i = \mathrm{id}$. $\square$

The main problem is thus to determine when the application η_H is surjective. In other words, if we have an extension $\mathcal{E}$ of $k\overline{N}_G(H)$-modules of V by W, can we find an extension of Mackey functors $\mathcal{F}$ of $T_{H,V}$ by $T_{H,W}$ whose evaluation at H equals $\mathcal{E}$? The next result tells us that, if such an extension exists, we have no choice for $\mathcal{F}$.

Proposition 3.2. *Let $\mathcal{E} : 0 \to W \xrightarrow{i} U \xrightarrow{q} V \to 0$ be an extension of $k\overline{N}_G(H)$-modules. If $\mathcal{F} : 0 \to T_{H,W} \xrightarrow{j} N \xrightarrow{p} T_{H,V} \to 0$ is an extension of Mackey functors whose evaluation at H equals $\mathcal{E}$, then $N \cong T_{H,U}$.*

Proof. If $U = 0$, the result is trivial. Assume that $U \neq 0$. If such an extension $\mathcal{F}$ exists, then H is a minimal subgroup of G such that $N(H) \neq 0$, and more precisely, $N(H) = U$. We are going to apply the criterion given by Proposition 2.5 in order to show that $N \cong T_{H,U}$. Let χ be the subconjugacy closure of H. We have to prove that $\mathrm{Ker}(R_\chi^N) = 0$ and that $\mathrm{Im}(I_\chi^N) = N$, where $\mathrm{Ker}(R_\chi^N)(K) = \bigcap_{X \in \chi,\, X \leqslant K} \mathrm{Ker}(R_X^K)$ and $\mathrm{Im}(I_\chi^N)(K) = \sum_{X \in \chi,\, X \leqslant K} \mathrm{Im}(I_X^K)$, for every $K \leqslant G$. Let us set $K \leqslant G$ such that there exists $g \in G$ with ${}^gH \leqslant K$ (otherwise $T_{H,W}(K) = 0 = T_{H,V}(K)$). We consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_{H,W}(K) \arrow[r, "j(K)"] \arrow[d, shift left, "R_1"] & N(K) \arrow[r, "p(K)"] \arrow[d, shift left, "R_2"] & T_{H,V}(K) \arrow[r] \arrow[d, shift left, "R_3"] & 0 \\ 0 \arrow[r] & \sum_{g \in T_G(H,K)} T_{H,W}({}^gH) \arrow[r, "j({}^gH)"] \arrow[u, shift left, "I_1"] & \sum_{g \in T_G(H,K)} N({}^gH) \arrow[r, "p({}^gH)"] \arrow[u, shift left, "I_2"] & \sum_{g \in T_G(H,K)} T_{H,V}({}^gH) \arrow[r] \arrow[u, shift left, "I_3"] & 0 \end{tikzcd}$$

where the applications I_i are given by $\sum_{g \in T_G(H,K)} I_{{}^gH}^K$ and the R_i by $\sum_{g \in T_G(H,K)} R_{{}^gH}^K$, for $i = 1, 2, 3$.

Using the properties of the functors $T_{H,W}$ and $T_{H,V}$, we obtain

$$\mathrm{Im}(I_1) = \mathrm{Im}\big(I_\chi^{T_{H,W}}\big)(K) = T_{H,W}(K)$$

and hence, I_1 is surjective. Similarly, I_3 is surjective. By the five Lemma, the application I_2 is also surjective. In a similar fashion, R_1 is injective, due to $\mathrm{Ker}(R_1) = \mathrm{Ker}(R_\chi^{T_{H,W}}) = 0$. As R_3 is also injective, R_2 is injective, still by the five Lemma. In other words, $\mathrm{Ker}(R_\chi^N) = 0$. We thus can conclude thanks to Proposition 2.5. $\square$

Using the previous result, we can describe the image of the application η_H in terms of the modules U and V:

Proposition 3.3. *Let H be a subgroup of G and let V and W be $k\overline{N}_G(H)$-modules. The group $\mathrm{Ext}^1_{\mu_k(G)}(T_{H,V}, T_{H,W})$ is isomorphic to the subgroup of extensions U of W by V such that $I_H^K(U) \cap W = I_H^K(W)$ for every subgroup $H \leqslant K \leqslant N_G(H)$.*

Proof. Using Eq. (3.0.1), we may assume that H is a normal subgroup. The evaluation morphism $\eta_H : \mathrm{Ext}^1_{\mu_k(N_G(H))}(T_{H,V}, T_{H,W}) \to \mathrm{Ext}^1_{k\overline{N}_G(H)}(V, W)$ is injective (see Proposition 3.1), thus we have to understand its image. By Proposition 3.2, this question is equivalent to the following problem: if $0 \to W \xrightarrow{i} U \xrightarrow{q} V \to 0$ is an exact sequence of kG/H-modules, when does an exact sequence of Mackey functors of type $0 \to T_{H,W} \xrightarrow{j} T_{H,U} \xrightarrow{p} T_{H,V} \to 0$ exist?

If such a sequence exists, then for any subgroup K with $H \leqslant K \leqslant G$, the application $j(K) : T_{H,W}(K) = I_H^K(W) \to T_{H,U}(K) = I_H^K(U)$ must satisfy

$$j(K)\big(I_H^K(w)\big) = I_H^K j(H)(w) = I_H^K i(w) = i\big(I_H^K(w)\big)$$

for every $w \in W$. Thus, $j(K)$ coincides with i, and similarly, $p(K)$ has to coincide with q. Moreover, it is easy to check that, with this definition, j is injective, p is surjective and $\mathrm{Im}(j) \subseteq \mathrm{Ker}(p)$. We then have to determine when $\mathrm{Ker}(p) \subseteq \mathrm{Im}(j)$. Yet, if $x \in \mathrm{Ker}(p(K))$, then $q(x) = 0$ and, as $\mathrm{Ker}(q) = \mathrm{Im}(i)$, it follows that $x \in I_H^K(U) \cap W$, for every $H \leqslant K \leqslant G$ (identifying W with $i(W)$ as i is injective). We can also identify $I_H^K(W)$ and $i(I_H^K(W))$ for every $H \leqslant K \leqslant G$. Consequently, $\mathrm{Ker}(p) \subseteq \mathrm{Im}(j)$ if and only if $I_H^K(U) \cap W = I_H^K(W)$ for every $H \leqslant K \leqslant G$. Hence, η_H is an isomorphism from $\mathrm{Ext}^1_{\mu_k(G)}(T_{H,V}, T_{H,W})$ onto the subgroup of extensions U of $\mathrm{Ext}^1_{k\overline{N}_G(H)}(V, W)$ such that $I_H^K(U) \cap W = I_H^K(W)$ for every subgroup $H \leqslant K \leqslant G$. $\square$

We are now able to deal with two different cases: the case where the modules that index the functors T are trivial (this case generalizes in particular the case of the p-groups) and the case where the group G possesses a normal Sylow p-subgroup:

Theorem 3.4. *Let H be a subgroup of a group G and k the trivial $k\overline{N}_G(H)$-module. If p is odd, then*

$$\mathrm{Ext}_{\mu_k(G)}(S_{H,k}, S_{H,k}) \cong \mathrm{Ext}_{k\overline{N}_G(H)}(k,k) \cong \mathrm{Hom}\big(\overline{N}_G(H), k^+\big)$$

where $\mathrm{Hom}(\overline{N}_G(H), k^+)$ is the k-vector space of group homomorphisms from $\overline{N}_G(H)$ to the additive group k^+.

If $p = 2$, then $\mathrm{Ext}_{\mu_k(G)}(S_{H,k}, S_{H,k})$ is isomorphic to the subgroup of extensions U of k by k, such that every element of order 2 of $\overline{N}_G(H)$ acts trivially on U. Consequently,

$$\mathrm{Ext}_{\mu_k(G)}(S_{H,k}, S_{H,k}) \cong \mathrm{Hom}\big(\overline{N}_G(H)/I_H, k^+\big)$$

where I_H is the subgroup of $\overline{N}_G(H)$ generated by involutions.

Proof. For the isomorphism between $\mathrm{Ext}_{k\overline{N}_G(H)}(k,k)$ and $\mathrm{Hom}(\overline{N}_G(H),k^+)$, see [2], Propositions 3.14.2 and 3.14.3.

By Proposition 3.3, $\mathrm{Ext}_{\mu_k(G)}(S_{H,k},S_{H,k})\cong \mathrm{Ext}_{k\overline{N}_G(H)}(k,k)$ if and only if for every subgroup K/H of $\overline{N}_G(H)$ and for every extension $0\to W\xrightarrow{i} U\xrightarrow{q} V\to 0$ of $k\overline{N}_G(H)$-modules, where $V\cong k\cong W$, we have $I_H^K(U)\cap W=I_H^K(W)$.

Let $H\leqslant K\leqslant N_G(H)$. If the order of the group K/H is prime to p, then $I_H^K(U)\cap W=U^{K/H}\cap W=W^{K/H}=I_H^K(W)$, as the application $I_H^K:U\to U^K$ is surjective. Suppose next that p divides the order of K/H. Since $W\cong k$, we deduce that $I_H^K(W)=[K:H]\cdot W=0$. Let $S/H\cong C_p$ be a minimal subgroup of K/H generated by an element g. The facts that $I_H^S(k)=0$ and that $U/W\cong V\cong k$ yield $I_H^S(U)\subseteq W$.

Assume that p is odd. If $x\in U$, then $g\bar{x}=\bar{x}$, where $\bar{x}$ is the image of x in U/W, which is isomorphic to the trivial module k. Hence, there exists an element $w\in W$ such that $gx=x+w$, and consequently,

$$I_H^S(x)=\sum_{i=0}^{p-1}g^ix=px+\sum_{i=0}^{p-1}iw=px+\frac{p(p-1)}{2}w=0.$$

It follows that $I_H^K(U)=I_S^K(I_H^S(U))=I_S^K(0)=0$, and hence,

$$I_H^K(U)\cap W=0=I_H^K(W).$$

On the one hand, if $p=2$ and if every element of order 2 of $N_G(H)/H$ acts trivially on U, then $gx=x$ for every $x\in U$, and, like before, $I_H^S(U)=0$. Thus, $I_H^K(U)=0$. On the other hand, if there exists an element $j\in\overline{N}_G(H)$ of order 2 and an element $x\in U$ such that $jx=x+w$ with $w\in W$, $w\neq 0$, then $I_H^J(x)=w\neq 0$ where $J=\langle j\rangle$. Consequently, $I_H^J(U)\cap W=I_H^J(U)\neq 0=I_H^J(W)$, and hence such an extension U does not originate from an extension of $S_{H,k}$ by $S_{H,k}$. $\square$

Theorem 3.5. *Let G be a group which has a normal Sylow p-subgroup, $H\leqslant G$, and let V and W be simple $k\overline{N}_G(H)$-modules.*

If p is odd, then $\mathrm{Ext}_{\mu_k(G)}(S_{H,V},S_{H,W})\cong \mathrm{Ext}_{k\overline{N}_G(H)}(V,W)$.

If $p=2$, then $\mathrm{Ext}_{\mu_k(G)}(S_{H,V},S_{H,W})$ is isomorphic to the subgroup of extensions U of V by W such that every element of order 2 of $\overline{N}_G(H)$ acts trivially on U.

Proof. Let us first remind that, by Eq. (3.0.1),

$$\mathrm{Ext}^1_{\mu_k(G)}\big(S^G_{H,V},S^G_{H,W}\big)=\mathrm{Ext}^1_{\mu_k(N_G(H))}\big(T^{N_G(H)}_{H,V},T^{N_G(H)}_{H,W}\big)$$

and since $\overline{N}_{N_G(H)}(H)=\overline{N}_G(H)$, we have $T^{N_G(H)}_{H,V}=S^{N_G(H)}_{H,V}$ and $T^{N_G(H)}_{H,W}=S^{N_G(H)}_{H,W}$, as the modules V and W are simple. We may then assume that H is a normal subgroup of G. Moreover, by Proposition 3.3, $\mathrm{Ext}(S_{H,V},S_{H,W})$ is isomorphic to the subgroup of $\mathrm{Ext}(V,W)$ of extensions U of V by W, such that $I_H^K(U)\cap W=I_H^K(W)$ for every subgroup K of G with $H\leqslant K$. Let us fix such an extension and a subgroup K of G containing H. We can then distinguish two cases:

(i) The order of K/H is divisible by p: in which case, let us fix a subgroup $J\cong C_p$ of K/H. The group $P/(P\cap H)\cong PH/H$ acts trivially on V and W, since it is normal in G/H (by the theorem of Clifford). Then, as J is a subgroup of PH/H, it has to act trivially on V and W. Hence, we may use the same proof as that of Theorem 3.4.

(ii) The order of K/H is prime to p: in which case, we are able to conclude as we did in the proof of Theorem 3.4, i.e. by using the fact the application I_H^K is surjective. $\square$

4. Extension groups of higher degree: An example

We will now focus on extension groups of higher degree in a framework where we know the structure of the indecomposable projective Mackey functors: the case of a group which has a Sylow p-subgroup of order p. Thévenaz and Webb indeed proved that $\mathrm{Mack}_k(G)$ has finite representation type if and only if p^2 does not divide the order of G and under this assumption, they explicitly described the structure of the indecomposable projective Mackey functors (see [7], Sections 18 and 20).

First of all, let us remark that the problem of determining extension groups of higher degree is equivalent to finding minimal projective resolution of simple Mackey functors. Indeed, $P_\star : \cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is a minimal projective resolution of a Mackey functor M if and only if $\mathrm{Hom}_A(P_n, S) \cong \mathrm{Ext}^n_A(M, S)$ for every $n > 0$ and every simple Mackey functor S. In other words, if $P_\star$ is a minimal projective resolution of M, then $P_i = \bigoplus_S (P_S)^{\dim_k(\mathrm{Ext}^i_A(M,S))}$, where the sum is taken over the simple Mackey functors S (up to isomorphism) and where P_S is the projective cover of S.

Given a group G with a Sylow p-subgroup C of order p, we are interested in determining the minimal projective resolution of simple Mackey functors. As there is an equivalence of categories between $\mathrm{Mack}_k(G, J)$ and $\mathrm{Mack}_k(\overline{N}_G(J), 1)$ (see [7], Theorem 10.1), we can work in $\mathrm{Mack}_k(G, 1)$ and thus merely deal with simple Mackey functors indexed by p-subgroups of G (which are 1 and C). Since our goal is to determine extension groups between two simple Mackey functors, we may take these functors inside the same block (otherwise the extension group is trivial).

We begin by studying the case where C is normal. In this case, C acts trivially on every simple kG-module V, hence every simple kG-module is also a simple kG/C-module. Moreover, blocks of $\mathrm{Mack}_k(G, 1)$ are in bijection with those of kG, and more precisely, two simple Mackey functors in $\mathrm{Mack}_k(G, 1)$ are in the same block if and only if the corresponding modules are in the same block of kG. Thévenaz and Webb explicitly described the structure of projective Mackey functors in this context (see [7], Theorem 20.1). Let us give a quick reminder of their Theorem: let B be a block of kG and $V_1, \ldots, V_e$ the simple modules in this block. These modules are all 1-dimensional, and e divides $p-1$. Moreover, for every $1 \leqslant i \leqslant e$, the projective cover P_{V_i} of the module V_i is uniserial and it is possible to index these modules so that the Loewy series of P_{V_i} equals:

$$P_{V_i} = \begin{matrix} V_i \\ V_{i+1} \\ \vdots \\ V_{i-1} \\ V_i \end{matrix}$$

with a cyclic ordering of the indexes i, where each V_j, for $j \neq i$, appears $\frac{p-1}{e}$ times and where V_i appears $\frac{p-1}{e} + 1$ times (for details, see [1], Sections 5 and 17). If b is the corresponding block of $\mathrm{Mack}_k(G, 1)$, then the simple Mackey functors of b are the S_{1,V_i} and S_{C,V_i} for $i = 1, \ldots, e$. The result of Thévenaz and Webb states that the Loewy series of the projective P_{C,V_i} is given by

$$P_{C,V_i} = \begin{matrix} S_{C,V_i} \\ S_{1,V_i} \\ S_{C,V_i} \end{matrix}.$$

Moreover, the projective functor P_{1,V_i} has a simple socle, isomorphic to S_{1,V_i}, and $\mathrm{Rad}(P_{1,V_i})/\mathrm{Soc}(P_{1,V_i}) \cong M_i \oplus S_{C,V_i}$, where M_i is an uniserial module whose layers of its Loewy series equal to $S_{1,V_{i+1}}, S_{1,V_{i+2}}, \ldots, S_{1,V_{i-1}}$, and each composition factor S_{1,V_j}, for $j \neq i$, appears $\frac{p-1}{e}$ times and S_{1,V_i} appears $\frac{p-1}{e} - 1$ times. In other words, the Loewy series of P_{1,V_i} equals:

$$P_{1,V_i} = \begin{array}{ccc} & S_{1,V_i} & \\ S_{1,V_{i+1}} & & S_{C,V_i} \\ S_{1,V_{i+2}} & & \\ \vdots & & \\ S_{1,V_{i-1}} & & \\ & S_{1,V_i} & \end{array}$$

We are now able to construct the minimal projective resolution of the simple Mackey functors indexed by the Sylow p-subgroup:

Proposition 4.1. *Let G be a group that has a normal Sylow p-subgroup C of order p, b a block of Mackey functors in $\mathrm{Mack}_k(G,1)$ and B the corresponding block of kG. Furthermore, let $V_1,\dots,V_e$ be the simple kG-modules in B, with a cyclic ordering of the indexes i. Finally, let us fix $1 \leqslant i \leqslant e$. The simple Mackey functor S_{C,V_i} possesses a minimal projective resolution which is given by*

$$\ldots \to B_{i+3} \to B_{i+2} \to B_{i+1} \to B_i \to S_{C,V_i} \to 0$$

where B_j equals $P_{C,V_{j+1}} \to P_{1,V_{j+1}} \to P_{1,V_j} \to P_{C,V_j}$ for every $j = 1,\dots,e$. In particular, this resolution is periodic of period $4e$.

Proof. The projective resolution can be explicitly constructed using the structure of the projective Mackey functors described above. For details, see [5, Proposition 3.2.6]. □

The previous proposition directly gives us extension groups of any degree between some simple Mackey functors:

Proposition 4.2. *Let G be a group that has a normal Sylow p-subgroup C of order p. Let b be a block of Mackey functors in $\mathrm{Mack}_k(G,1)$ and B the corresponding block of kG. Let $V_1,\dots,V_e$ be the simple kG-modules in B, with a cyclic ordering of the indexes. Then*

$$\mathrm{Ext}^n_{\mu_k(G)}(S_{C,V_i}, S_{C,V_j}) = \begin{cases} k & \text{if } n-4(j-i) \equiv 0, -1 \bmod 4e, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{Ext}^n_{\mu_k(G)}(S_{C,V_i}, S_{1,V_j}) = \begin{cases} k & \text{if } n-4(j-i) \equiv -2, 1 \bmod 4e, \\ 0 & \text{otherwise.} \end{cases}$$

We can now use Proposition 4.1 to show that if G is a group which has a Sylow p-subgroup C of order p (not necessarily normal), then every simple Mackey functor for G indexed by C possesses a minimal projective resolution which is periodic.

Corollary 4.3. *Let G be a group that has a Sylow p-subgroup C of order p and V a simple $k\overline{N}_G(C)$-module. The simple Mackey functor $S^G_{C,V}$ then has a minimal projective resolution that is periodic, of period dividing $4e$, where e is the number of simple modules that lie in the same block of $N_G(C)$ as V.*

Proof. Using the definition of the functor $S^G_{C,V}$, we obtain that $S^G_{C,V} = S^{N_G(C)}_{C,V}\uparrow^G_{N_G(C)}$. First of all, by Proposition 4.1, the minimal projective resolution $(P_\star)$ of the functor $S^{N_G(C)}_{C,V}$ is periodic, of period $4e$, where e is the number of simple modules which lie in the same block of $N_G(C)$ as the module V. Next, we apply the induction functor from $N_G(C)$ to G to the resolution $(P_\star)$. As the induction functor is exact (see [8], Proposition 4.2) and as induction preserves projectives (see Proposition 1.4), we get a projective resolution of $S^G_{C,V}$ which is periodic. So, we deduce that the minimal projective resolution of $S^G_{C,V}$ is also periodic of period dividing $4e$. □

We will now focus on the simple Mackey functors indexed by the trivial subgroup for the group $C \rtimes C_e$, where $C \cong C_p$, where C_e acts faithfully on C and where e divides $p-1$. Let us stress the fact that this case is fundamental in the frame of group algebras kG, where G is a group which has a Sylow p-subgroup of order p. Indeed, one can show that if B is a block of the algebra $kN_G(C)$, then B is Morita-equivalent to the category of the $k(C \rtimes C_e)$-modules, and the simple or projective modules are preserved by this equivalence (see [2], Proposition 6.5.4 and Section 2.2).

If $G = C \rtimes C_e$, then the group algebra kG has only one block containing all the simple kG-modules which are also simple kG/C-modules. Since $G/C \cong C_e$, there are exactly e simple modules, $V_1, \ldots, V_e$, all of dimension 1. More precisely, if we let ζ be a e^{th} primitive root of unity and g a generator of C_e, then $V_i = kx_i$ and the action of g is given by $gx_i = \zeta^i x_i$ for every $i = 1, \ldots, e$, with a cyclic ordering of the indexes. Consequently, $\mathrm{Mack}_k(G,1)$ contains $2e$ simple Mackey functors: S_{1,V_i} and S_{C,V_i} for every $i = 1, \ldots, e$.

Proposition 4.4. *Let $G = C \rtimes C_e$, where $C \cong C_p$, where C_e acts faithfully on C and where e divides $p-1$. Moreover, let $V_1, \ldots, V_e$ be the simple kG-modules, with a cyclic ordering of the indexes. Finally, let us fix $1 \leqslant i \leqslant e$. The simple Mackey functor S_{1,V_i} then has the following minimal projective resolution*

$$\ldots \to B_{i+3} \to B_{i+2} \to B_{i+1} \to B_i \to S_{1,V_i} \to 0,$$

where B_j equals $P_{1,V_{j+1}} \to P_{C,V_{j+1}} \oplus P_{1,V_j} \to P_{C,V_j} \oplus P_{1,V_{j+1}} \to P_{1,V_j}$ for every $j = 1, \ldots, e$. In particular, this resolution is periodic, of period $4e$.

Proof. The first thing to do is to construct minimal projective resolution of simple Mackey functor $S_{1,k}$ for the cyclic group C of order p. We can then induce the resolution from C to G and we get the result, using Proposition 1.4, Theorem 1.3 and Proposition 4.2. For details, see [5], Proposition 3.2.9. □

Again, we can state the corresponding result about the extension groups.

Proposition 4.5. *Let $G = C \rtimes C_e$, where $C \cong C_p$, where C_e acts faithfully on C and where e divides $p-1$. Let $V_1, \ldots, V_e$ be the simple kG-modules with a cyclic ordering of the indexes. Then*

$$\mathrm{Ext}^n_{\mu_k(G)}(S_{1,V_i}, S_{1,V_j}) = \begin{cases} k & \text{if } n - 4(j-i) \equiv 1, 3, 4, 6 \bmod 4e, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{Ext}^n_{\mu_k(G)}(S_{1,V_i}, S_{C,V_j}) = \begin{cases} k & \text{if } n - 4(j-i) \equiv -2, 1 \bmod 4e, \\ 0 & \text{otherwise.} \end{cases}$$

As for the case of simple Mackey functors indexed by the Sylow p-subgroup, we can show that, if G is a group with a Sylow p-subgroup C of order p, then every simple Mackey for G indexed by the trivial subgroup has a periodic minimal projective resolution. Nevertheless, the situation is more complicated since one cannot see $S_{1,V}$ as induced from a Mackey functor for the group $C \rtimes C_e$, where e divides $p-1$. However, we will see that there exists an indecomposable Mackey functors M for C such that $S_{1,V}$ is a direct summand of $M{\uparrow}_C^G$. We will thus begin by showing that every indecomposable Mackey functors for the cyclic group of order p has a minimal projective resolution which is periodic:

Proposition 4.6. *Let $P = C_p$ be the cyclic group of order p and M an indecomposable non-projective Mackey functor for P. The minimal projective resolution of M is periodic, of period 4 (possibly after a few steps).*

Proof. We use the list of indecomposable Mackey functors for P explicitly constructed in the proof of Theorem 18.1 of [7]. We then construct their minimal projective resolution by direct calculation (for details, see [5], Proposition 3.2.11). □

Proposition 4.7. *Let G be a group with a Sylow p-subgroup C of order p and V a simple non-projective kG-module. The Mackey functor $S_{1,V}^G$ possesses then a minimal projective resolution which is periodic.*

Remark. If the module V is simple and projective, then the Mackey functor $S_{1,V}$ is also projective (see [7], Corollary 17.3) and consequently, the minimal projective resolution of $S_{1,V}$ is trivial.

Proof. Proposition 11.4 of [7] states that C is a vertex of $S_{1,V}$ and that $T_{1,U}^C$ is a source of $S_{1,V}$, where U is a source of V. Thus, $S_{1,V}$ is a direct summand of $T_{1,U}^C\uparrow_C^G$, which is an indecomposable Mackey functor for C (see Proposition 2.8). Hence, thanks once again to Proposition 4.6, we know that it has a minimal projective resolution which is periodic. As induction is an exact functor (see [8], Proposition 4.2) and as induction preserves projective Mackey functors (see Proposition 1.4), the functor $T_{1,U}^C\uparrow_C^G$ possesses a periodic projective resolution, hence its minimal projective resolution is also periodic. Finally, as $S_{1,V}$ is a direct summand of $T_{1,U}^C\uparrow_C^G$, its minimal projective resolution has to be periodic. □

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